

Extended-body effects in cosmological spacetimes

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Abstract. We study the dynamics of extended test bodies in flat Friedmann-Robertson-Walker spacetimes. It is shown that such objects can usually alter their inertial mass, spin, and center-of-mass trajectory purely through the use of internal deformations. Many of these effects do not have Newtonian analogs, and exist despite the presence of conserved momenta associated with the translational and rotational symmetries of the background.

1. Introduction

It is well-known that sufficiently small objects tend to fall along the geodesics of whichever spacetime they inhabit. While it is difficult to precisely state the limits of this result (see e.g. [1, 2, 3, 4, 5, 6]), some qualifications are clearly necessary. Even in Newtonian gravity, the details of an extended test body’s internal structure can cause its center-of-mass trajectory to diverge from that of an equivalent point particle. Inhomogeneities in the external field effectively couple to the higher multipole moments of the mass distribution to modify the center-of-mass motion.

In principle, this effect allows piloted spacecraft to partially modify their trajectories simply by rearranging internal masses. A particularly elegant example of this is a strategy whereby artificial satellites can change their orbital parameters by cleverly manipulating tethered masses [7]. Different parts of the body effectively “push” or “pull” on local gradients in the gravitational field. In the relativistic context, test bodies interact with the background spacetime using their full stress-energy tensors rather than just their mass distributions. The control space available to alter trajectories using extended body effects is therefore greatly enlarged. This has a number of interesting consequences if very large stresses and internal momenta can be maintained.

It allows, for example, a test body which starts at rest in a homogeneous (but nonstationary) spacetime to accelerate purely through the use of internal manipulations. No rocketry of any sort is required. Even though there is no field anisotropy for the body to push off of, it may still control its trajectory to some degree. This is because a spacetime which may appear instantaneously uniform to an observer comoving with the center-of-mass line needn’t have this property for the various frames associated with

each of the body’s constituents. Sufficiently large internal velocities therefore allow a temporal asymmetry in the geometry to have an effect very similar to a spatial one. Related to this is an ability for an initially nonrotating body to spin itself up by reducing its center-of-mass velocity (appropriately defined).

There are other qualitatively new effects which occur in the relativistic theory as well. Among these is the fact that a body’s rest mass is usually not conserved. This may be traced to changes in what may be interpreted as the energy of the system. The “gravitational potential energy” can vary with a body’s shape or the nature of the applied field, for example. This affects its mass.

We show how to make these claims precise, and illustrate them using the example of an uncharged extended test body inhabiting a spatially-flat Friedmann-Robertson-Walker (FRW) spacetime. This provides one of the first specific treatments of higher order finite-size effects in the literature. Most previous work has considered fully relativistic extended test bodies only as spinning point particles [24, 25, 26, 27]. These results assumed that quadrupole and higher order effects were negligible, although this is not always physically reasonable. There are many systems in which quadrupole effects dominate over those associated with a particle’s spin. While the general forms of these corrections have been considered by several authors [10, 11, 12, 13, 14, 16, 17], they do not appear to have been previously studied in any specific systems.

2. Laws of motion

The constructions used here derive from an extensive formalism developed by Dixon [15, 16, 17, 18] and others [19, 20, 21] to generalize useful concepts in Newtonian gravitational mechanics to curved spacetimes. It considers the behaviour of an extended body described by a nonsingular stress-energy tensor T^{ab} with spatially-bounded support W . To briefly summarize, it has been shown that it is possible to convert (without any approximation) the laws of motion

$$\nabla_a T^{ab} = 0 \tag{1}$$

into a handful of ordinary differential equations. These act on objects which may be interpreted as the body’s linear and angular momenta. Simply put, a clear separation is made between those components of T^{ab} which are affected by stress-energy conservation, and those which are purely constitutive. These latter quantities may be identified as the body’s higher multipole moments (starting at the quadrupole). A number of other phenomenologically interesting quantities naturally arise in this formalism as well. These may be interpreted as a body’s inertial mass, gravitational potential energy, and so on.

For the present purposes, Dixon’s formalism is most useful when supplemented with a center-of-mass definition. Under mild assumptions, it is possible to define a worldline Γ which may be shown to be uniquely defined, timelike, and within the convex hull of W [20]. It also reduces to the standard center-of-mass definition in the appropriate limits. Rather than solving for the full stress-energy tensor throughout the worldtube, we only track Γ and a few supplementary quantities necessary to compute it.

Before proceeding with this, it is first necessary to define a body's linear and angular momenta. Taking cues from the standard flat-spacetime definitions of these quantities [22, 23], there should be some sense in which the momenta depend on time. At any particular instant, it is reasonable to define them by integrating the body's stress-energy tensor over an appropriate spacelike hypersurface. To be more specific, suppose that the center-of-mass worldline is parameterized by some function $\gamma(s)$. s then defines a natural notion of time. A spacelike hypersurface $\Sigma(s)$ which contains $\gamma(s)$ may now be associated with each point on Γ . The set of all such hypersurfaces will be assumed to foliate the body's worldtube.

With these constructions in place, note that a particle's linear momentum should be a vector field. In particular, let $p^a(s)$ be a vector at $\gamma(s)$. Similarly, the angular momentum $S^{ab}(s)$ is taken to be a rank-2 skew-symmetric tensor at $\gamma(s)$. There are an infinite number of reasonable definitions for momenta with these properties. A unique choice may be made by supposing that any Killing vector which a spacetime may possess implies that some linear combination of p^a and S^{ab} is conserved. It is clear that any Killing vector ξ^a will imply the existence of a conserved quantity

$$C_{(\xi)} = \int_{\Sigma(s)} d\Sigma_a (\xi_b T^{ab}) . \quad (2)$$

We now suppose that $C_{(\xi)}$ may be constructed from the momenta according to

$$C_{(\xi)} = p^a(s) \xi_a(\gamma(s)) + \frac{1}{2} S^{ab}(s) \nabla_a \xi_b(\gamma(s)) . \quad (3)$$

This uniquely defines $p^a(s)$ and $S^{ab}(s)$ in terms of $T^{ab}(x)$, $\gamma(s)$, and $\Sigma(s)$. The resulting expressions are completely independent of ξ^a , so they may be adopted even in the absence of any symmetries. Their detailed forms are not needed here, and may be found in [16, 18].

Temporarily assuming that there again exists a Killing vector ξ^a , (3) indicates that some linear combination of the force $F^a(s)$ and torque $N^{ab}(s) = N^{[ab]}(s)$ should vanish. Differentiating (3) with respect to s , one easily finds that

$$F^a \xi_a + \frac{1}{2} N^{ab} \nabla_a \xi_b = 0 , \quad (4)$$

where

$$F^a := \delta p^a / ds - \frac{1}{2} S^{bc} v^d R_{bcd}{}^a \quad (5)$$

$$N^{ab} := \delta S^{ab} / ds - 2p^{[a} v^{b]} , \quad (6)$$

and $v^a(s) = \dot{\gamma}^a(s) = \delta \gamma^a / ds$. Note that the force automatically excludes the ‘‘pole-dipole’’ component of $\delta p^a / ds$, while the definition of the torque removes the term responsible for Thomas precession. Both (5) and (6) make sense even in the absence of any Killing vectors, so we consider these definitions to be general.

Expressions for the force and torque may be derived from (1) in terms of T^{ab} [17, 18]. If the spacetime geometry inside each slice $W \cap \Sigma(s)$ is sufficiently smooth (as discussed more precisely in [15]), the resulting equations may be expanded as asymptotic series

involving successively higher multipole moments of the stress-energy tensor. To lowest non-vanishing (i.e. quadrupole) order, it may be shown that [16, 17, 18]

$$F_a(s) \simeq -\frac{1}{6} J^{bcd\mathfrak{f}}(s) \nabla_a R_{bcd\mathfrak{f}}(\gamma(s)) \quad (7)$$

$$N^{ab}(s) \simeq \frac{4}{3} J^{cd\mathfrak{f}[a}(s) R^{b]}_{\mathfrak{f}cd}(\gamma(s)) . \quad (8)$$

$J^{abcd} = J^{[ab][cd]} = J^{cdab}$ is the quadrupole moment of the stress-energy tensor. It is defined precisely in [17], although we simply note here that it does not satisfy any differential equations as a consequence of (1). All of the higher multipole moments are constructed such that they share this property. Their evolution equations are almost completely free, and depend only on the type of matter under consideration.

Up until now, we have been assuming that $\gamma(s)$ and its associated hypersurfaces are known. These will now be chosen uniquely using “center-of-mass” conditions. First let $n^a(s)$ be a unit timelike future-directed vector field defined on Γ . In a sense, the hypersurfaces $\Sigma(s)$ are chosen to essentially be hyperplanes orthogonal to $n^a(s)$. Their precise definition may be found in [16, 17, 18]. Regardless, our first center-of-mass condition demands that the linear momentum be proportional to this new vector field:

$$p^a = m n^a . \quad (9)$$

$m(s) > 0$ is some (not necessarily constant) proportionality factor which we interpret as the object’s inertial mass. This relation suggests that n^a be called the body’s “dynamical velocity.” This is generally distinct from the “kinematical velocity” v^a .

Next, suppose that

$$p_a S^{ab} = 0 . \quad (10)$$

This reduces to the standard center-of-mass condition in flat spacetime. It also leaves the angular momentum tensor with only three independent components. Indeed, S^{ab} may now be written in terms of a single vector S^a satisfying $n_a S^a = 0$:

$$S^{ab} = \epsilon^{abcd} n_c S_d . \quad (11)$$

Under mild assumptions, (9) and (10) uniquely determine n^a and Γ throughout any given object [20]. We call the worldline obtained with this method the center-of-mass line.

Writing down equations of motion in the standard way now requires an evolution equation for $\gamma(s)$. Despite the highly implicit nature of the center-of-mass relations, it is possible to derive an exact expression for $v^a - n^a$. Before describing this, it will first be convenient to fix the scale of s such that $v^a n_a = -1$. In general, the time parameter used here therefore fails to be the proper time of an observer moving on Γ . It is often very close, however.

Defining the left and right duals of the Riemann tensor as

$$R^*_{abcd} := \frac{1}{2} \epsilon^{pq}_{cd} R_{abpq} , \quad {}^*R_{abcd} := \frac{1}{2} \epsilon_{ab}{}^{pq} R_{pqcd} , \quad (12)$$

it can now be shown that [19]

$$(m^2 + {}^*R_{bcdf}^* n^b S^c n^d S^f) (n^a - v^a + \tau^a) = \epsilon^{abpq} n_p S_q [F_b - R_{bcdf}^* (n^c + \tau^c) n^d S^f] . \quad (13)$$

τ^a is that component of the torque which appears in the expansion

$$N^{ab} = \epsilon^{abcd} n_c N_d + 2m\tau^{[a} n^{b]} . \quad (14)$$

This decomposition is unique if $n_a N^a = n_a \tau^a = 0$.

The quadrupole moment J^{abcd} can also be written in terms of simpler tensor fields. Let [19]

$$J^{abcd} = S^{abcd} - n^{[a} \pi^{b]cd} - n^{[c} \pi^{d]ab} - 3n^{[a} Q^{b][c} n^{d]} . \quad (15)$$

$Q^{ab}(s) = Q^{(ab)}(s)$ may be interpreted as the quadrupole moment of the mass distribution seen by an observer at $\gamma(s)$ with velocity $n^a(s)$. Similarly, $\pi^{abc} = \pi^{a[bc]}$ and $S^{abcd} = S^{[ab][cd]} = S^{cdab}$ are essentially the body's momentum and stress quadrupoles. These objects satisfy

$$n_a Q^{ab} = n_b \pi^{abc} = n_a S^{abcd} = 0 , \quad (16)$$

and

$$\pi^{[abc]} = 0 . \quad (17)$$

There are therefore 8 independent components of π^{abc} . Q^{ab} and S^{abcd} each contain 6 components, leaving 20 for the full quadrupole moment of the stress-energy tensor. This total could also have been deduced immediately by noting that J^{abcd} shares the same algebraic symmetries as the Riemann tensor.

The equations discussed in this section form the complete laws of motion given by Dixon in the quadrupole approximation. They are not the only such equations in the literature, however. Various alternatives have been proposed [10, 11, 12, 13, 14], although each of these has drawbacks described in the introduction of [17]. To summarize, some contain calculational errors or inconsistent approximations. Others do not include the “full” quadrupole moments of the stress-energy tensor. Rather, some notions of the momentum and stress quadrupoles are (explicitly or implicitly) neglected. Some also imply undesirable results even in flat spacetime: e.g. changing masses or accelerating center-of-mass worldlines. These issues were all resolved by Dixon's formalism. For this reason and others, it will therefore be adopted universally for the remainder of this article.

3. Conservation laws

The previous section has developed enough of Dixon's formalism to allow the study of test body motion in the quadrupole approximation. The spatially-flat FRW spacetimes we consider admit at least six linearly independent Killing vectors. Each of these implies the existence of its own conserved quantity, which leaves four degrees of freedom which

are free to affect the evolution of Γ . Suppose that we choose coordinates such that the metric takes the form

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) . \quad (18)$$

The scale factor $a(t)$ will be considered given. If Einstein's equation is assumed to hold, it depends on the averaged density and pressure of the background matter in the standard way.

Generators of the isometries associated with this FRW background may be written down by inspection. First define the orthonormal tetrad frame

$$\mathbf{e}_{(0)} = \partial_t , \quad \mathbf{e}_{(i)} = a^{-1} \partial_i , \quad (19)$$

where $i = 1, 2, 3$. Using a 3-dimensional “vector” notation to denote triads, the minimal set of Killing vector fields associated with (18) may now be written in the form

$$\vec{\xi} = a\vec{e} , \quad \vec{\omega} = \vec{R} \times \vec{\xi} . \quad (20)$$

Here, \vec{R} has been defined by the triplet (x, y, z) . The cross product notation used to write down the rotational symmetries $\vec{\omega}$ then has the standard interpretation. For example, rotations about the z -axis are generated by the vector field $\omega_{(z)} = x\partial_y - y\partial_x$.

If (18) is assumed to hold throughout W , each of these six Killing vectors implies the existence of a conserved quantity of the form (2). Neglecting the body's self-field essentially defines the test body approximation as it is used here, although this assumption is not trivial. At first glance, it would seem to imply that the cosmological fluid penetrates the body, and that T^{ab} is negligible compared to the stress-energy tensor of the background. Most interesting systems do not satisfy these constraints. Intuitively, though, they shouldn't have to. This problem is a standard one, and we shall not attempt to address it directly. It will instead be assumed that the methods used here can be rigorously justified for a wide range of reasonable objects.

With this caveat out of the way, the constants associated with the background symmetries – usually referred to as the system's conserved momenta – may be directly related to Dixon's analogs of these quantities. Slightly abusing the notation, the collection of scalars associated with $\vec{\xi}$ through (2) will be referred to as the body's global linear momentum $\vec{P} = (C_{(\xi_{(x)})}, C_{(\xi_{(y)})}, C_{(\xi_{(z)})})$. In contrast, p^a may be thought of as the system's *local* linear momentum. Directly evaluating (3) using the translational Killing vectors given in (20) provides a direct relation between these two objects. Simplifying it using (9), (11), and the various normalization and orthogonality conditions shows that

$$\vec{P} = a(m\vec{n} - H\vec{n} \times \vec{S}) . \quad (21)$$

Here, \vec{n} and \vec{S} denote the triad components of the appropriate vector fields with respect to \vec{e} . As is standard, the Hubble parameter H in (21) is defined by

$$H := \frac{d \ln a}{dt} . \quad (22)$$

The system's two notions of angular momenta may be understood in a very similar way. Let those constants associated the rotational Killing vectors $\vec{\omega}$ be denoted by \vec{L} .

As before, a direct evaluation of (3) immediately relates this object to p^a , S^a , and γ . After simplifying with (21), it reduces to

$$\vec{L} - \vec{\gamma} \times \vec{P} = \frac{(1 + |\vec{n}|^2)\vec{S} - (\vec{n} \cdot \vec{S})\vec{n}}{(1 + |\vec{n}|^2)^{1/2}}. \quad (23)$$

The dot product notation used here is just the standard Euclidean definition. The norm $|\vec{n}|^2$ is then $\vec{n} \cdot \vec{n}$. In deriving (23), the Killing vectors must be evaluated at the center-of-mass position $\gamma(s)$. We have therefore replaced \vec{R} by $\vec{\gamma} = (\gamma^x, \gamma^y, \gamma^z)$. Notational conventions aside, the left-hand side of (23) may naturally be interpreted as the “spin component” of the global angular momentum. In most cases, it differs very little from the local spin angular momentum \vec{S} . Note, however, that it does not necessarily remain conserved. \vec{P} and \vec{L} are always fixed. $\vec{\gamma}$ is not.

4. Forces and torques

We now have explicit relations between the global and local notions of momentum. Using the background tetrad, they may be used to write n^a and S^a in terms of γ , m , H , \vec{P} , and \vec{L} . This does not uniquely fix a body’s center-of-mass position, however. Its velocity v^a satisfies (13), which also requires knowledge of the force and torque. In the FRW spacetimes considered here, N^{ab} is entirely determined by the vector field τ^a defined in (14). With the exception of the temporal component $F^a \nabla_a t$, this quantity also fixes the force.

These statements follow from considering the consequences of the conservation laws implied by the spacetime’s symmetries. As explained in Sec. 2, each of these implies that a particular linear combination of the force and torque must vanish. In this case, there are therefore six relations between the various components of F^a and N^{ab} which may be written down without any detailed knowledge of the system. Directly applying (4) for each of the Killing vectors in (20) shows that they have the form

$$\vec{F} = \frac{mH\vec{\tau}}{\sqrt{1 + |\vec{n}|^2}} \quad (24)$$

and

$$\vec{N} = \frac{m(\vec{n} \times \vec{\tau})}{\sqrt{1 + |\vec{n}|^2}}. \quad (25)$$

There remain four free parameters here: $\vec{\tau}$ and $F^a \nabla_a t$. These depend on the body’s internal structure, and are essentially free. There exists some extended body which may be associated with almost any imaginable pair. The only restrictions are imposed by energy conditions and the consistency of the test body approximation.

In general, specific relations between a body’s internal structure and the force and torque acting on it may be found using the exact integral expressions in [17, 18]. If the body under consideration is much smaller than a Hubble length, its force and torque may instead be approximated using multipole expansions. Although the conservation laws used to derive (24) and (25) are valid in the exact theory, they remain exactly

satisfied in any multipole approximation [19]. It is therefore fully consistent to consider the global momenta fixed while using the quadrupole expressions for the force and torque given in Sect. 2.

5. Converting linear to rotational motion

The two conservation laws (21) and (23) may be used to write \vec{n} and \vec{S} in terms of \vec{P} and $\vec{L} - \vec{\gamma} \times \vec{P}$. The first of these quantities is always conserved, while the second is not. $\vec{\gamma} \times \vec{P}$ may vary as the body moves, which implies that the “global spin” may change. This suggests that moving bodies are able to change their local spin S^a by appropriately manipulating their internal structures.

Roughly speaking, the dynamics of the background spacetime mixes the momenta. \vec{P} depends on both \vec{S} and \vec{n} , for example. This implies that if \vec{n} or $\vec{p} = m\vec{n}$ can be changed, so will \vec{S} . Taking the norm of (21) illustrates this directly:

$$|\vec{p}| = \frac{|\vec{P}/a|}{\sqrt{1 + |H\vec{S}_\perp/m|^2}} . \quad (26)$$

Here, \vec{S}_\perp represents the component of \vec{S} orthogonal to \vec{n} . A larger spin may therefore be obtained by decreasing an object’s (local) linear momentum.

This equation only takes into account the conservation of linear momentum, so one might expect (23) to rule out any such effects. It does not. Suppose for simplicity that $\vec{P} \cdot \vec{L} = 0$, but $\vec{P} \neq 0$. Then $\vec{n} \cdot \vec{S} = 0$ regardless of the particle’s internal dynamics. It follows that

$$\vec{S} = \vec{S}_\perp = \frac{\vec{L} - \vec{\gamma} \times \vec{P}}{\sqrt{1 + |\vec{n}|^2}} . \quad (27)$$

In this case, the magnitude of the angular momentum vector coincides with that of the triad components displayed here: $S^a S_a = \vec{S} \cdot \vec{S}$. This clarifies the interpretation of (26).

Combining (26) with (27) allows either $|\vec{S}|$ or $|\vec{n}|$ to be expressed in terms of m , H , $|\vec{P}/a|$, and $|\vec{L} - \vec{\gamma} \times \vec{P}|$. Although straightforward to derive, the resulting expressions are quite lengthy. We therefore note only that to lowest order in \vec{P} , they show that

$$|\vec{S}_\perp| = |\vec{L} - \vec{\gamma} \times \vec{P}| \left[1 + o(|\vec{P}/ma|^2) \right] . \quad (28)$$

This can also be deduced directly by noting that $|\vec{n}| \simeq |\vec{P}/ma|$ in this limit. The denominator of (27) may therefore be expanded directly in the global momentum. (28) follows trivially.

Regardless, this illustrates that the magnitude of a given particle’s spin depends on its center-of-mass position. In the simplest cases, $d\vec{\gamma}/ds$ will be nearly proportional to \vec{P} , implying that $|\vec{S}_\perp|$ remains constant. More generally, though, extended-body effects allow significant variations in the spin.

It is interesting to ask whether these changes are generic. Consider a particle which does not rotate, so $\vec{S} = 0$. In this case, (21) and (23) imply that

$$\vec{\gamma} \times \vec{P} = \vec{L} , \quad \vec{n} = \vec{P}/ma . \quad (29)$$

Differentiating the first of these expressions with respect to s shows that $\vec{v} \times \vec{n} = 0$. This condition is both necessary and sufficient to ensure that the local angular momentum vanishes. \vec{v} must therefore remain proportional to \vec{n} . Using (13) finally shows that $\vec{\tau} \propto \vec{P}$. As might have been expected directly from (25), a nonspinning particle can only remain in that state if its internal structure is arranged such that the temporal torque remains proportional to the conserved momentum. Other possibilities alter the body's direction of travel, and inevitably impart a spin in the process.

This statement assumes that \vec{S} initially vanishes. Cases where $\vec{\tau} \propto \vec{n}$ always have special significance, however (at least when $\vec{P} \cdot \vec{L} = 0$). Differentiating (27) directly, it is clear that $d\vec{S}_\perp/ds$ involves terms proportional to both \vec{S}_\perp and $\vec{v} \times \vec{P}$. The second of these is more interesting.

The center-of-mass velocity may be found using (13) and (24). Here, choosing $\tau \propto \vec{n}$ implies that

$$n^a - v^a + \tau^a \propto -\epsilon^{abpq} n_p S_q R_{bcd}^* n^c n^d S^f . \quad (30)$$

It may then be shown that $\vec{v} \propto \vec{n}$. Computing the spin therefore requires knowledge of $\vec{n} \times \vec{P}$. This is most easily obtained by using (21) and (23) to show that

$$am\vec{n} = \frac{\vec{P} + (H/m)\vec{P} \times \vec{S}_\perp}{1 + |H\vec{S}_\perp/m|^2} . \quad (31)$$

It follows that whenever $\vec{\tau} \propto \vec{n}$,

$$d\vec{S}_\perp/ds \propto \vec{S}_\perp . \quad (32)$$

The direction (but not necessarily the magnitude) of the local angular momentum therefore remains fixed in all such cases.

6. Zero-momentum particles

The detailed behaviour of extended bodies with arbitrary momenta can be quite complicated. Suppose for simplicity that

$$\vec{P} = 0 . \quad (33)$$

This is the unique condition which locks a particle's motion to that of the background fluid in the monopole and dipole approximations. Interesting effects appear at the quadrupole level, and are surprisingly rich even in this simple case.

The spin behaviour discussed in the previous section is not retained, however. Dotting (21) with \vec{n} implies that \vec{p} must vanish whenever \vec{P} does. It then follows from (23) that $\vec{S} = \vec{L}$ in this case. The local and global angular momenta agree exactly, so the local spin is conserved. The choice (33) therefore isolates those extended body effects which are as independent of the angular momentum as possible.

Contracting (13) with $e_{(0)}^a$ and recalling that $n_a \tau^a = 0$ now shows that up to a possible additive constant, the worldline parameter s is simply the cosmological

comoving time t . The remainder of (13) then yields the equation-of-motion

$$m \frac{d\vec{\gamma}}{dt} = \frac{m\vec{\tau} + H \left[\vec{L} \times \vec{\tau} + (H/m)(\vec{L} \cdot \vec{\tau})\vec{L} \right]}{a(1 + |H\vec{L}/m|^2)} . \quad (34)$$

This expression reduces to $\vec{v} = a d\vec{\gamma}/dt = \vec{\tau}$ when $\vec{L} \propto \vec{\tau}$. In other words, the coordinate velocity matches the coordinate (not frame) components of τ^a . Unless the angular momentum is extremely large, this will remain an excellent approximation for the center-of-mass velocity of any system satisfying $\vec{P} = 0$.

The temporal torque $\vec{\tau}$ – and therefore the velocity \vec{v} – may be controlled by a body allowed to manipulate its internal structure. Combining (14), (15), and (8), it can be shown that in the quadrupole approximation,

$$m\vec{\tau} \simeq \frac{2}{3} \dot{H} \vec{\Pi} . \quad (35)$$

Here, $\dot{H} := dH/dt = dH/ds$ and

$$\Pi^a := \pi^{ba}{}_{,b} . \quad (36)$$

This trace of the momentum quadrupole may point in any direction whatsoever (orthogonal to n^a), so an object with vanishing global momentum could have complete control over its direction of motion. The magnitude of this quantity limits the magnitude of the center-of-mass velocity. Indeed,

$$|\vec{v}| \simeq |\vec{\tau}| \left(\frac{1 + (H/m)^2 (\vec{L} \cdot \vec{\tau})^2 / |\vec{\tau}|^2}{1 + |H\vec{L}/m|^2} \right)^{1/2} . \quad (37)$$

It follows that $|\vec{v}| \lesssim |\vec{\tau}|$ in the quadrupole approximation. These two quantities coincide only when $\vec{S} \times \vec{\Pi} = 0$ or $\vec{S} \cdot \vec{\Pi} = 0$. In other cases, a nonzero spin will reduce the magnitude of the particle’s “drift” velocity.

$\vec{\tau}$ depends on the mass parameter m , so \vec{v} does as well. In general, (5) and (9) may be used to show that

$$\frac{dm}{ds} = R_{abcd}^* n^a v^b n^c S^d - n_a F^a , \quad (38)$$

Adapted to the present case, this reduces to

$$\frac{dm}{dt} \simeq -\frac{1}{2} \frac{d}{dt} (\dot{H} + H^2) Q^a{}_a + \frac{1}{3} \frac{d}{dt} (H^2) S^{ab}{}_{ab} . \quad (39)$$

Mass changes therefore depend only on the quadrupole moments of the body’s mass and stress distributions. They are independent of the momentum quadrupole. This quantity only affects the particle’s velocity with respect to the cosmological rest frame, which is itself independent of Q^{ab} and S^{abcd} . Each of these three components of the full quadrupole moment may be varied independently. By construction, stress-energy conservation does not require any coupling between them. In principle, a properly-engineered spacecraft could therefore control its mass and velocity separately.

The coefficients in front of the moments in (39) are written so as to make this equation trivial to integrate if the traces $Q^a{}_a$ and $S^{ab}{}_{ab}$ remain constant. That this is

possible is not an accident. In the exact theory, the mass may be written as a sum of terms which are naturally interpreted as the total internal energy m_{int} , the gravitational potential energy Φ , and the rotational kinetic energy E_{rot} of the body [18]:

$$m = m_{\text{int}} + \Phi + E_{\text{rot}} . \quad (40)$$

The rotational contribution here may be defined in terms of S^a and an angular velocity derived from an inertia tensor based on Q^{ab} [16, 18]. If the quadrupole moment remains fixed in an appropriate sense, the fact that $S_a S^a = |\vec{L}|^2$ cannot change ensures that E_{rot} remains constant.

More generally, it is probably most straightforward to study the behaviour of the sum $m_{\text{int}} + E_{\text{rot}} = m - \Phi$. This may be found by noting that in the quadrupole approximation, the gravitational potential energy reduces to [16]

$$\Phi \simeq \Phi_0 + \frac{1}{6} J^{abcd} R_{abcd} , \quad (41)$$

where Φ_0 is an arbitrary constant. Applying this to the present case, one sees that

$$\Phi \simeq \Phi_0 - \frac{1}{2}(\dot{H} + H^2)Q^a{}_a + \frac{1}{3}H^2 S^{ab}{}_{ab} . \quad (42)$$

Comparison with (39) now shows that the internal and rotational energies must evolve according to

$$\frac{d(m - \Phi)}{dt} = \frac{1}{2}(\dot{H} + H^2)\frac{dQ^a{}_a}{dt} - \frac{1}{3}H^2\frac{dS^{ab}{}_{ab}}{dt} . \quad (43)$$

This effectively describes the rate at which energy is inductively absorbed from the gravitational field.

It has been proven in the exact theory that m_{int} remains constant for any “rigid” body; i.e. one whose multipole moments do not change with respect to an appropriate co-rotating tetrad [18]. In this special case, full rigidity is not required even of the quadrupole moment. The internal energy is conserved for any object where $Q^a{}_a$ and $S^{ab}{}_{ab}$ remain constant. This condition is weaker than one requiring that all tetrad components of J^{abcd} remain constant.

When evaluating (37) and (39) in specific spacetimes, it is useful write \dot{H} and \ddot{H} in terms of quantities with more direct physical interpretations. For the metric (18), Einstein’s equation is equivalent to [23]

$$3H^2 = 8\pi\rho + \Lambda \quad (44)$$

$$3\ddot{a}/a = -4\pi(\rho + 3p) + \Lambda . \quad (45)$$

ρ and p are respectively the density and pressure of the background matter, while Λ represents the cosmological constant. Suppose for simplicity that the cosmological fluid obeys an equation of state of the form

$$p = w\rho . \quad (46)$$

Also introduce the density parameter

$$\Omega := 8\pi\rho/3H^2 = 1 - \Lambda/3H^2 . \quad (47)$$

Ignoring the particle's spin, (37) and (39) now reduce to

$$m\vec{v} \simeq -(1+w)\Omega H^2 \vec{\Pi} , \quad (48)$$

and

$$\dot{m} \simeq -(1+w)\Omega H^3 \left[\frac{3}{4}(1+3w)Q^a{}_a + S^{ab}{}_{ab} \right] . \quad (49)$$

If Ω vanishes or $w = -1$, the spacetime is de Sitter. In these cases, both an object's mass and position remain fixed for all time regardless of its internal structure. This could have been deduced directly using the fact that de Sitter spacetimes possess an extra four Killing vectors beyond the six in (20). Together, these ensure that all components of the force and torque must vanish. There exist ten conserved quantities, among which is m . The center-of-mass line is always a geodesic in this case, and the angular momentum will be parallel-propagated along that geodesic.

Generically, though, the object's mass will almost always change. Nearly any imaginable object has a nonzero mass quadrupole. Standard energy conditions should then imply that $Q^a{}_a \neq 0$. Excluding the de Sitter case, the first term in (49) can only vanish if $w = -1/3$. This possibility only marginally satisfies the strong energy condition, so it is unlikely to be physically relevant.

Still, these effects are exceedingly small even in the most favorably-chosen systems. If the characteristic diameter of W in the center-of-mass frame is of order $D(s)$, the maximum $|\vec{v}|$ which the particle may attain by the method we've described is of order $(D/10^{10} \text{ yr})^2$ in the present universe. This value can only become significant for objects on the scale of galactic superclusters. It is a best-case estimate, however. The relatively low internal velocities of most types of astrophysical matter makes it very unlikely that there exists any observable drift velocity even in these systems.

The situation improves slightly if we no longer assume that $\vec{P} = 0$. For small momenta, \vec{v} can be shown to depend on the mass quadrupole as well as $\vec{\Pi}$. This eliminates the need for a complicated internal velocity distribution. All that's required is an appropriate mass distribution together with some bulk motion. In either case, though, the maximum change in the center-of-mass velocity still scales as (characteristic speed) $\times (D/10^{10} \text{ yr})^2$. The ultra-relativistic limit $|\vec{P}|/ma \gg 1$ is more interesting. If the magnitude of J^{abcd} is kept fixed with respect to a center-of-mass frame, boosting the quadrupole moment appropriately leads to a considerable amplification of the effects we've discussed. Of course, this case isn't particularly relevant to astrophysical systems either.

Mass shifts are potentially more interesting. Although the fractional *rate* of mass change will be very small, no special structure is required to maintain it (and its sign). For very large objects, the total change in m over a significant fraction of the universe's history could be significant. Suppose that $Q^a{}_a$ remains constant, and ignore the stress quadrupole. (39) and (44) then imply that in a matter-dominated universe, an object's mass will change by an amount

$$m_f - m_i \simeq -\frac{1}{4}Q^a{}_a H_f^2 \left[(a_f/a_i)^3 - 1 \right] . \quad (50)$$

Given that this scales as the cube of the redshift factor between the initial and final states, relatively large changes may exist even though $Q^a_a H_f^2 \ll m$. For example, a galactic supercluster ($D \sim 10^8$ yr today) formed at a redshift of ~ 10 might have lost as much as 5% of its mass by the present time. Of course, this model is very crude. Among other problems, there is no reason to expect that Q^a_a would remain even approximately constant for so long. It is also not clear how the notion of mass used here – which is essentially inertial – relates to the gravitational masses so fundamental to astronomical observations. At least in stationary spacetimes, these two types of mass are not identical [21]. Understanding the analogous relations in the present case would require further investigation.

Mass loss effects have been noted before in cosmological spacetimes for freely-falling point particles endowed with a scalar charge [8, 9]. In that case, the effect is due to an object’s own scalar radiation reflecting back towards itself. This arises from the failure of Huygen’s principle in curved spacetimes. Such exotic mechanisms are apparently not required to induce mass changes in freely-falling particles (even rigid ones).

7. Discussion

Despite the simplicity of the systems studied here, a number of interesting extended-body effects were found to exist. Even in the presence of linear and angular momentum conservation laws, it was shown that bodies can control the magnitudes of their center-of-mass acceleration and spin using purely internal processes. This illustrates some of the richness of finite-size effects in general relativity. Similar phenomena surely exist in other spacetimes, and their magnitudes are likely to be much larger for objects with realistic dimensions.

It’s also worthwhile to note that the concept of controllable motion emphasized here doesn’t occur in simpler approximation schemes. If the quadrupole and higher multipole moments of a body are ignored in its laws of motion, one recovers the Papapetrou equations (occasionally called the Mathisson-Papapetrou or Mathisson-Papapetrou-Dixon equations). These follow from (1), and involve no free parameters once a definition is fixed for the center-of-mass. The motion of a spinning test body in a given background is completely determined by its initial conditions.

This no longer remains true once the effects of higher multipole moments are taken into account. Dixon has defined these moments in such a way that stress-energy conservation does not affect their evolution. Their time-dependence is made unique by specifying constitutive relations which depend on the type of material under consideration. These are essentially equations of state, and are simplest to consider in the case of an ideal programmed mechanical device. The internal structure may then be considered a given function of time.

Applying the equations of motion we’ve derived to more standard “passive” materials can be relatively complicated. This requires specifying an evolution equation for the quadrupole moment in terms of the various other bulk quantities relevant to

the motion. For most realistic materials, all of the multipole moments couple to each other. It may not always be possible to ignore these couplings, in which case the body is probably best modeled using the full partial differential equations of continuum mechanics.

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